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The Helmholtz conditions revisited. A new approach to the inverse problem of Lagrangian dynamics

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Abstract. This paper deals with the general problem of finding a multiplier matrix which can give to a prescribed system of second-order ordinary differential equations the structure of Euler–Lagrange equations. The approach is based on a generalisation of previous studies on linear systems. We also pay attention to the relationship with a recently proposed alternative procedure, with respect to which we gain considerable simplifications and new insights. The main result concerns a set of necessary and sufficient conditions for the existence of a multiplier, which contains an infinite set of algebraic equations, the coefficients of which can be used to derive necessary conditions involving only the given right-hand sides of the differential equations. An outline is given of interesting points for future studies, and an example is presented for which all multipliers are explicitly constructed.

1. Introduction

The inverse problem of Lagrangian mechanics, or more generally the inverse problem of the calculus of variations, is a subject which has been studied over several decades, and in fact traces back to the previous century. It concerns for instance the question under what circumstances a given system of second-order ordinary differential equations

$$\ddot{q}^i = f^i(t, q, \dot{q}) \qquad i = 1, \ldots, n \tag{1}$$

can be derived from a variational principle, or more precisely, under what conditions a non-singular multiplier matrix $(\alpha_{ij}(t, q, \dot{q}))$ can be constructed such that

$$\alpha_{ij}(\ddot{q}^j - f^j) \equiv \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} \tag{2}$$

for some function $L(t, q, \dot{q})$. The necessary and sufficient conditions under which a general expression like $\alpha_{ij}(t, q, \dot{q})\ddot{q}^j + \beta_i(t, q, \dot{q})$ can be identified with the right-hand side of (2) are usually referred to as the Helmholtz conditions (Helmholtz 1887), although Helmholtz did not prove their sufficiency. They are mostly written as

$$\alpha_{ij} = \alpha_{ji} \tag{3}$$

$$\frac{\partial \alpha_{ij}}{\partial \dot{q}^k} = \frac{\partial \alpha_{ik}}{\partial \dot{q}^j} \tag{4}$$

$$\frac{\partial \beta_i}{\partial \dot{q}^j}+\frac{\partial \beta_j}{\partial \dot{q}^i}=2\left(\frac{\partial}{\partial t}+\dot{q}^k\frac{\partial}{\partial q^k}\right)\alpha_{ij} \tag{5}$$

$$\frac{\partial \beta_i}{\partial q^j}-\frac{\partial \beta_j}{\partial q^i}=\frac{1}{2}\left(\frac{\partial}{\partial t}+\dot{q}^k\frac{\partial}{\partial q^k}\right)\left(\frac{\partial \beta_i}{\partial \dot{q}^j}-\frac{\partial \beta_j}{\partial \dot{q}^i}\right). \tag{6}$$

Despite the long-term interest in this problem and the numerous contributions, one can readily say that nobody has ever come close to solving the general problem (apart from some very recent developments quoted later on). The reason for this opinion is that there are many different ways of deriving the Helmholtz conditions, and many of the authors in some sense are doing just that. In an enumeration of possible approaches (and sample papers in each area) we can mention the so-called self-adjointness of the equations (Santilli 1978), the introduction of an appropriate type of differential calculus (Dedecker 1950, Tulczyjew 1975, Lawruk and Tulczyjew 1977), the use of a Vainberg-type theory of potential operators (Tonti 1969a, b, Vanderbauwhede 1978, 1979), or a more abstract geometrical approach making use of cohomology theory (Takens 1979). It is by no means our intention to criticise such contributions, which are sometimes of extreme importance in providing better insight into the nature of the problem. To name only some merits; certain approaches succeed in deriving Helmholtz-type conditions in one and the same formula for differential equations of arbitrary order or for the case of field equations; others set the stage for a global discussion of variational calculus. In a purely local context, however, and for second-order systems like (1), the hard problem is to try to eliminate the multiplier α from the Helmholtz conditions, in other words to come up with conditions for the existence of a multiplier, which are expressed in terms of the given functions f^i only. Two cases have been solved this way in the older literature. First of all, there is the rather trivial case of one degree of freedom, for which Darboux (1894) already proved that a Lagrangian always exists, and for which various nice properties can be discussed (see e.g. Currie and Saletan 1966, Kobussen 1979, Sarlet 1981). Secondly, Douglas (1941) has completely solved the case $n=2$. The complexity of his analysis has probably discouraged people for a long time from trying further generalisations.

Over the last few years now, there has been an intensive study of the inverse problem for linear second-order equations (Bahar *et al* 1978, Kwatny *et al* 1979, Sarlet 1980, Novak and Milic 1980, Sarlet *et al* 1982), apparently for its possible relevance for applications in network theory (Bahar and Kwatny 1980). The contributions of Sarlet *et al* (1982) may be said to come close to the solution for general linear systems, because they provide an infinite set of algebraic necessary and sufficient conditions, the consistency of which will inevitably lead to vanishing determinants, which then exactly constitute conditions on the given elements of the equations of motion. This analysis moreover gave rise to a number of interesting sufficiency criteria for the existence of a Lagrangian.

It is the purpose of the present paper to extend the analysis for linear systems to the general problem formulated by (2). This will in the first place require reworking the Helmholtz conditions (3)–(6), bringing them into a simplified and equivalent form, much along the guidelines set out for linear systems. The first theorem in that procedure has already been derived in a different way by Douglas (1941). Part of the further reconditioning has been independently derived by Novak (1981). From the reduced form of the Helmholtz conditions, we are again able to produce an infinite set of necessary and sufficient conditions, which are all algebraic in nature, except for two. These two exceptions of course embody the much greater complexity of the general problem with respect to the case of linear systems. The algebraic-type conditions constitute a simplification of similar conditions recently derived by Henneaux (1981), who made use of a quite elegant procedure invoking the translation of the Helmholtz conditions to a set of conditions on the equivalent first-order system in q and $\dot{q}$. We report on Henneaux' approach in an appendix, and provide details about its relationship with the present approach. In the final sections of the paper we give an outline of interesting features for future study, we relate our approach to Douglas' work for $n=2$, and we present an example.

2. Other forms of the Helmholtz conditions

Our starting point will be the conditions (3)–(6), in which in accordance with (2) we set

$$\beta_i = -\alpha_{ij} f^j. \tag{7}$$

We further introduce the notations

$$A_{ij}(t, q, \dot{q}) = -\frac{1}{2}\frac{\partial f^i}{\partial \dot{q}^j} \qquad B_{ij}(t, q, \dot{q}) = -\frac{\partial f^i}{\partial q^j} \tag{8}$$

$$\beta_{ij} = \frac{1}{2}\left(\frac{\partial \beta_j}{\partial \dot{q}^i} - \frac{\partial \beta_i}{\partial \dot{q}^j}\right) \Rightarrow \beta = A^{\mathrm{T}}\alpha - \alpha A \tag{9a, b}$$

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \dot{q}^k \frac{\partial}{\partial q^k} + f^k \frac{\partial}{\partial \dot{q}^k}. \tag{10}$$

It is then straightforward to verify that the equations (5) and (6) can be rewritten as

$$\frac{\mathrm{D}\alpha}{\mathrm{D}t} = \alpha A + A^{\mathrm{T}}\alpha \qquad \frac{\mathrm{D}\beta}{\mathrm{D}t} = B^{\mathrm{T}}\alpha - \alpha B \tag{11a, b}$$

whereby one will have to use the identity

$$\frac{\partial \beta_{ij}}{\partial \dot{q}^k} \equiv \frac{\partial \alpha_{jk}}{\partial q^i} - \frac{\partial \alpha_{ik}}{\partial q^j}. \tag{12}$$

Moreover, the matrix β can be completely eliminated from the picture (in view of (9b)), which leads to the following result.

Theorem 1. A non-singular $n \times n$ matrix $\alpha(t, q, \dot{q})$ will be a multiplier for (1) if and only if it satisfies the conditions

$$\alpha = \alpha^{\mathrm{T}} \qquad \frac{\partial \alpha_{ij}}{\partial \dot{q}^k} = \frac{\partial \alpha_{ik}}{\partial \dot{q}^j} \tag{13a, b}$$

$$\frac{\mathrm{D}\alpha}{\mathrm{D}t} = \alpha A + A^{\mathrm{T}}\alpha \tag{13c}$$

$$\alpha\Phi^{(0)} = (\alpha\Phi^{(0)})^{\mathrm{T}} \tag{13d}$$

where $\Phi^{(0)}(t, q, \dot{q})$ is a matrix defined by

$$\Phi^{(0)} = B - A^2 - \mathrm{D}A/\mathrm{D}t. \tag{14}$$

Remarks. This theorem was already derived by Douglas (1941), who proved the necessity and sufficiency of conditions (13) without passing through the Helmholtz conditions (3)–(6).

The notations A, B and $\Phi^{(0)}$ (and similar quantities later on) are chosen in such a way that they coincide for linear systems with the same quantities in Sarlet *et al* (1982).

At this point, it is interesting to make a connection with an alternative approach, which starts from the self-adjointness conditions of first-order equations. Consider a general system of first-order equations like

$$\dot{x}^{\mu} = F^{\mu}(t, x) \qquad \mu = 1, \ldots, 2n. \tag{15}$$

Then it is well known (see e.g. Sarlet and Cantrijn 1978) that a non-singular multiplier matrix $C_{\mu\nu}(t, x)$ will make (15) derivable from a variational principle, if and only if it is skew symmetric and satisfies

$$\frac{\partial C_{\mu\nu}}{\partial x^{\alpha}} + \frac{\partial C_{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial C_{\alpha\mu}}{\partial x^{\nu}} = 0 \tag{16}$$

$$\frac{\partial C_{\mu\nu}}{\partial t} = \frac{\partial}{\partial x^{\mu}}(C_{\nu\alpha}F^{\alpha}) - \frac{\partial}{\partial x^{\nu}}(C_{\mu\alpha}F^{\alpha}). \tag{17}$$

The conditions (16) characterise a closed 2-form in x-space, depending parametrically on t (alternatively, (16) and (17) define a closed 2-form in (t, x)-space). For further use, we shall refer to (16) as the closure conditions. Consider now the case that (15) is equivalent to the second-order system (1), i.e. let

$$x = \mathrm{col}(q, \dot{q}) \qquad F = \mathrm{col}(\dot{q}, f) \tag{18}$$

then the existence of a Lagrangian multiplier can be translated to a first-order counterpart by the following result.

Theorem 2. Equation (1) will have a multiplier α if and only if the corresponding first-order system defined through (18) and (15) has a multiplier $(C_{\mu\nu})$, satisfying the additional restriction

$$C_{\mu\nu} = 0 \qquad \text{for } \mu, \nu = n+1, \ldots, 2n. \tag{19}$$

More details about this transition can be found in Sarlet (1979). The same theorem has recently been derived in more geometrical terms by de Ritis *et al* (1981), by Henneaux (1981) and by Crampin (1981).

From now on we assume that all given functions are analytic in a certain domain Ω of (t, x) space. Then, (17) is a partial differential equation for $C_{\mu\nu}$, which has been solved in terms of the derivative with respect to one specific variable (t). Hence, it has the form to which the Cauchy–Kowalewski theorem (see e.g. Courant and Hilbert 1962, p 39, Haack and Wendland 1972, p 15) can be applied, ensuring the existence and uniqueness of analytic solutions in terms of arbitrarily assigned 'initial functions' $C_{\mu\nu}(t_0, x)$. Since $\partial/\partial t$ of the left-hand side of (16) is identically zero in view of (17), it is obvious that a solution of (17) will satisfy (16) at all t, as soon as it satisfies it at $t = t_0$. Henneaux' procedure exploits the fact that a similar property does not hold concerning the additional algebraic conditions (19). We show in the appendix that the first few steps in Henneaux' method allow one to conclude that C must have the form

$$C=\begin{pmatrix}\beta & -\alpha\\ \alpha & 0\end{pmatrix} \qquad \alpha=\alpha^{\mathrm{T}} \qquad \beta=-\beta^{\mathrm{T}}. \tag{20}$$

The evolution-type equations (17), when written out for the separate parts α and β in C, precisely yield equations (11). According to theorem 2, we then still have to impose the closure conditions (16). For a C of type (20), they give rise to three equations. One of these is just equation (13*b*). Another one appears to be the identity (12). The third one reads

$$\frac{\partial\beta_{ij}}{\partial q^k}+\frac{\partial\beta_{jk}}{\partial q^i}+\frac{\partial\beta_{ki}}{\partial q^j}=0. \tag{21}$$

Since we have not encountered this condition in the context of theorem 1, it has to be an identity too (see the appendix). Note that this redundancy in the number of closure conditions can also be given a geometrical interpretation. Indeed, the reader who consults Crampin's recent contribution (Crampin 1981) will discover a connection with the fact that the closedness of the 2-form ω (whose coefficient matrix is C) can be replaced by some weaker conditions in terms of Lagrangian subspaces (which, by the way, also explains the appearance of the zero-block in C).

3. Further reduction of the Helmholtz conditions

For linear systems, we succeeded in making a further reduction, consisting just in replacing condition (13*a*) by a similar requirement on the initial value only. This may seem to be a rather unimportant detail, but it is indispensable for completing the derivation of the infinite set of algebraic conditions, which can ultimately be used to obtain requirements on the f^i only.

So, it is quite natural that we try a similar reduction in the present general situation, not only concerning condition (13*a*), but also with respect to the additional condition (13*b*). The following lemma will be useful for that purpose.

Lemma 3. Let $g:R\times R^m\to R^k$, $(t,x)\to g(t,x)$ (m and k arbitrary) satisfy a partial differential equation of the type

$$\partial g/\partial t=a_j(t,x)\partial g/\partial x^j+G(t,x,g) \qquad g(t_0,x)=0 \tag{22}$$

with coefficients which are analytic in some suitable domain. Assume further that

$$\partial^l G(t_0,x,0)/\partial t^l=0 \qquad \forall l=0,1,\ldots \tag{23}$$

for all x in the related domain. Then $g(t,x)\equiv 0$.

Proof. In view of the analyticity, assumptions (23) imply $G(t,x,0)=0$. This in turn means that (22) has the zero solution. Since the Cauchy–Kowalewski theorem ensures uniqueness of the solution with given Cauchy data, the conclusion follows.

Applying the above lemma to the equation

$$\mathrm{D}(\alpha-\alpha^{\mathrm{T}})/\mathrm{D}t=(\alpha-\alpha^{\mathrm{T}})A+A^{\mathrm{T}}(\alpha-\alpha^{\mathrm{T}})$$

which follows from (13*c*), it immediately follows that the condition (13*a*) can be replaced by

$$\alpha(t_0, q, \dot{q}) = (\alpha(t_0, q, \dot{q}))^{\mathrm{T}}.$$

Concerning condition (13*b*), one could proceed in exactly the same way: first find out what differential equation will be satisfied by that condition as a result of (13*c*), and then continue this procedure for newly introduced terms until a closed system is obtained to which lemma 3 can be applied. One then would come to the conclusion that it is not enough to assume (13*b*) at $t = t_0$ only. Quite remarkably, however, it is sufficient to add as an assumption the next term in the Taylor expansion of (13*b*) at $t = t_0$, namely

$$\frac{\partial^2 \alpha_{ij}}{\partial t \partial \dot{q}^k}(t_0, q, \dot{q}) = \frac{\partial^2 \alpha_{ik}}{\partial t \partial \dot{q}^j}(t_0, q, \dot{q}). \tag{24}$$

Proving this statement becomes somewhat easier if one knows about the equivalent 'first-order picture' as described in the previous section. To be precise, we shall show now that under the assumptions of theorem 1, but with (13*b*) replaced by (13*a*) at $t = t_0$ and (24), all closure conditions (16) hold true at $t = t_0$. We then know that they are equally true for all t, so that in particular (13*b*) will be satisfied for all t.

First recall that (13*c*) and (13*d*) imply (11*b*) with β defined by (9*b*). If we again introduce the vector components β_i by (7), (9*a*) is no longer valid for all t, but the present assumptions ensure that

$$\beta_{ij}(t_0) = \frac{1}{2}\left(\frac{\partial \beta_j}{\partial \dot{q}^i} - \frac{\partial \beta_i}{\partial \dot{q}^j}\right)(t_0) \qquad \frac{\partial \beta_{ij}}{\partial t}(t_0) = \frac{1}{2}\left(\frac{\partial^2 \beta_j}{\partial t \partial \dot{q}^i} - \frac{\partial^2 \beta_i}{\partial t \partial \dot{q}^j}\right)(t_0). \tag{25a, b}$$

(*Note*: here and in what follows, for any function $g(t, q, \dot{q})$, we shall abbreviate $g(t_0, q, \dot{q})$ to $g(t_0)$.)

Next, by using (25*a*), the differential equation (13*c*) for α, and the present assumptions at $t = t_0$, one arrives at

$$\frac{\partial \beta_{ij}}{\partial \dot{q}^k}(t_0) = \left(\frac{\partial \alpha_{jk}}{\partial q^i} - \frac{\partial \alpha_{ik}}{\partial q^j}\right)(t_0) \tag{26}$$

which is one of the closure conditions (16) at $t = t_0$.

Making use of (26), the differential equation (11*b*) for β produces the property

$$\left[\left(\frac{\partial}{\partial t} + \dot{q}^l \frac{\partial}{\partial q^l}\right)\beta_{ij}\right](t_0) = \left(\frac{\partial \beta_j}{\partial q^i} - \frac{\partial \beta_i}{\partial q^j}\right)(t_0). \tag{27}$$

Then, from (25*a*), proceeding as for (A3), and using (27) and (25), one easily derives the third closure condition (21) at $t = t_0$, which completes the proof.

When we think of applying the Cauchy–Kowalewski theorem to (13*c*), with t as the independent variable which is singled out, the conditions (24) do not have an appropriate form because they do not impose restrictions on the initial functions $\alpha(t_0, q, \dot{q})$. They can, however, equivalently be replaced by the closure condition (26) at $t = t_0$, and this condition can entirely be expressed in terms of $\alpha(t_0)$. Collecting together all the previous results we have thus proved the following theorem.

Theorem 4. A non-singular matrix $\alpha(t, q, \dot{q})$ will be a multiplier for (1) if and only if it satisfies

$$\alpha(t_0) = \alpha(t_0)^{\mathrm{T}} \qquad \frac{\partial \alpha_{ij}}{\partial \dot{q}^k}(t_0) = \frac{\partial \alpha_{ik}}{\partial \dot{q}^j}(t_0) \tag{28a, b}$$

$$\left[\frac{\partial \alpha_{ki}}{\partial q^j} - \frac{\partial \alpha_{ji}}{\partial q^k} + \frac{1}{2}\frac{\partial}{\partial \dot{q}^i}\left(\frac{\partial f^l}{\partial \dot{q}^j}\alpha_{lk}\right) - \frac{1}{2}\frac{\partial}{\partial \dot{q}^i}\left(\frac{\partial f^l}{\partial \dot{q}^k}\alpha_{lj}\right)\right](t_0) = 0 \tag{28c}$$

$$\frac{\mathrm{D}\alpha}{\mathrm{D}t} = \alpha A + A^{\mathrm{T}}\alpha \qquad \alpha\Phi^{(0)} = (\alpha\Phi^{(0)})^{\mathrm{T}}. \tag{28d, e}$$

Remark. It is worthwhile commenting on this result in connection with Henneaux' procedure as outlined in the appendix. We have already seen in theorem 1 that the closure conditions (16) can be simplified to only one condition, namely (13b), when they are imposed for all t. We now see that passing to conditions on the initial functions at $t = t_0$ requires the introduction of a second closure condition, namely (26) at $t = t_0$, but we still have a simplification with respect to (16) because (21) is still an identity at $t = t_0$ if the other conditions hold true.

The next theorem states a further reformulation of theorem 4 which, for linear systems, has a natural interpretation as being related to a 'reduction to canonical form' (see Sarlet *et al* 1982).

Theorem 5. Let $U(t, q, \dot{q})$ be the non-singular solution of the matrix partial differential equation

$$\mathrm{D}U/\mathrm{D}t + AU = 0 \tag{29a}$$

with initial value $U(t_0, q, \dot{q}) = 1$, the $n \times n$ unit matrix (solution guaranteed by the Cauchy–Kowalewski theorem). Then every multiplier α of (1) is of the form

$$\alpha = (U^{-1})^{\mathrm{T}} S U^{-1} \tag{29b}$$

and the necessary and sufficient conditions to be satisfied by $S(t, q, \dot{q})$ read

$$S(t_0) = S(t_0)^{\mathrm{T}} \qquad \frac{\partial S_{ij}}{\partial \dot{q}^k}(t_0) = \frac{\partial S_{ik}}{\partial \dot{q}^j}(t_0) \tag{29c, d}$$

$$\left[\frac{\partial S_{ki}}{\partial q^j} - \frac{\partial S_{ji}}{\partial q^k} + \frac{1}{2}\frac{\partial}{\partial \dot{q}^i}\left(\frac{\partial f^l}{\partial \dot{q}^j}S_{lk} - \frac{\partial f^l}{\partial \dot{q}^k}S_{lj}\right)\right](t_0) = 0 \tag{29e}$$

$$\mathrm{D}S/\mathrm{D}t = 0 \qquad SZ = (SZ)^{\mathrm{T}} \tag{29f, g}$$

$Z(t, q, \dot{q})$ being defined by

$$Z = U^{-1}\Phi^{(0)}U. \tag{30}$$

Proof. U being uniquely defined by (29a) and its initial value, (29b) can be considered as a substitution which defines S in terms of α or vice versa. Taking the total time derivative of (29b), and making use of the property

$$\frac{\mathrm{D}U^{-1}}{\mathrm{D}t} = -U^{-1}\frac{\mathrm{D}U}{\mathrm{D}t}U^{-1}$$

it is straightforward to verify that α will satisfy (28d) if and only if S satisfies (29f).

Equally simple is the verification that (29g), with Z defined by (30), is equivalent to (28e). Finally, in view of the initial value of U we have

$$\alpha(t_0, q, \dot{q}) = S(t_0, q, \dot{q}) \tag{31}$$

so that (29c)–(29e) immediately follow from (28a)–(28c) and vice versa.

Theorem 5 is in fact a refinement of a similar result, independently derived by Novak (1981), whereby the difference lies essentially in our introduction of the initial value t_0 in (29c)–(29e), seemingly a detail, but indispensible for the full translation of conditions on α to conditions on S, and for the further analysis in § 4. It is already possible to draw some general conclusions from the results in this section, such as the corollaries presented below, the numbering of which refers to the statement from which they follow.

Corollary 1.1. If the given functions f^i are independent of the velocities, then a multiplier which would not depend on $\dot{q}$ either, necessarily has to be constant.

Proof. Looking at theorem 1, the present assumptions imply $A = 0$, and therefore α has to be a matrix of constants of the motion. First integrals which would not depend on $\dot{q}$ can of course only be trivial constants.

A similar statement for the special case of diagonal multipliers was made by Havas (1957).

Corollary 1.2. System (1) is of Lagrangian type in its given form if and only if

$$A = -A^{\mathrm{T}} \qquad \Phi^{(0)} = \Phi^{(0)\mathrm{T}}. \tag{32}$$

Proof. Conditions (32) are precisely necessary and sufficient for (13) to have the unit matrix as solution for α.

Note: the matrix U becomes orthogonal when A is skew symmetric.

Corollary 5.1. Suppose that (1) has two different multipliers α_1 and α_2, and put

$$M = \alpha_1 \alpha_2^{-1}.$$

Then (i) det(M) is a first integral of (1), (ii) the traces of all powers of M are first integrals.

Proof. From theorem 5 we know that

$$\alpha_1 = (U^{-1})^{\mathrm{T}} S_1 U^{-1} \qquad \alpha_2 = (U^{-1})^{\mathrm{T}} S_2 U^{-1}$$

where all elements of S_1 and S_2 are first integrals. The result then follows trivially.

Property (i) was first derived by Lutzky (1979); property (ii) was recently derived at length by Hojman and Harleston (1981).

4. An infinite set of necessary and sufficient conditions

Now we go back to theorem 5, and proceed further along the lines followed for linear systems by Sarlet *et al* (1982). Computing the total time derivative of the matrix Z

defined by (30), and taking account of (29a), we obtain

$$\frac{\mathrm{D}Z}{\mathrm{D}t} = U^{-1}\Phi^{(1)}U \tag{33}$$

with

$$\Phi^{(1)}(t, q, \dot{q}) = \frac{\mathrm{D}\Phi^{(0)}}{\mathrm{D}t} + [A, \Phi^{(0)}]$$
$$[A, \Phi^{(0)}] = A\Phi^{(0)} - \Phi^{(0)}A. \tag{34}$$

Similarly we define

$$\Phi^{(k+1)}(t, q, \dot{q}) = \frac{\mathrm{D}\Phi^{(k)}}{\mathrm{D}t} + [A, \Phi^{(k)}]. \tag{35}$$

Writing $Z^{[k]}$ for the kth-order total time derivative of Z, we then have

$$Z^{[k]} = U^{-1}\Phi^{(k)}U. \tag{36}$$

All matrices $\Phi^{(k)}$ are expressions involving the given f^i only. They could also in a natural way be brought into the picture by proceeding with conditions (28) instead of (29) (see Sarlet 1980). In fact, since the two matrices α and S must coincide at $t = t_0$, it will be convenient to introduce a new notation for their common initial value,

$$\gamma(q, \dot{q}) = \alpha(t_0, q, \dot{q}) = S(t_0, q, \dot{q}), \tag{37}$$

so that the following main theorem can then be attached to either theorem 4 or theorem 5.

Theorem 6. Equation (1) will have a multiplier α if and only if a non-singular matrix $\gamma(q, \dot{q})$ exists, satisfying

$$\gamma = \gamma^{\mathrm{T}} \qquad \frac{\partial \gamma_{ij}}{\partial \dot{q}^k} = \frac{\partial \gamma_{ik}}{\partial \dot{q}^j} \tag{38a, b}$$

$$\frac{\partial \gamma_{ki}}{\partial q^j} - \frac{\partial \gamma_{ji}}{\partial q^k} + \frac{1}{2}\frac{\partial}{\partial \dot{q}^i}\left(\frac{\partial f^l}{\partial \dot{q}^j}\gamma_{lk} - \frac{\partial f^l}{\partial \dot{q}^k}\gamma_{lj}\right) = 0 \tag{38c}$$

$$\gamma\Phi^{(k)}(t_0) = (\gamma\Phi^{(k)}(t_0))^{\mathrm{T}} \qquad k = 0, 1, \ldots, \infty \tag{39(k)}$$

the matrices $\Phi^{(k)}(t, q, \dot{q})$ being recursively defined by (35).

Proof. (i) *Necessity.* Assuming a multiplier exists, equations (29) will have a solution for S. From (29f) and (29g) it then follows by taking successive derivatives that

$$SZ^{[k]} = (SZ^{[k]})^{\mathrm{T}} \qquad k = 0, 1, \ldots, \infty.$$

Observing from (36) that

$$Z^{[k]}(t_0, q, \dot{q}) = \Phi^{(k)}(t_0, q, \dot{q}) \tag{40}$$

leads to the desired result.

(ii) *Sufficiency.* Assume that γ satisfies (38) and (39), and let $S(t, q, \dot{q})$ be the solution of (29f) with γ as the initial value (by Cauchy–Kowalewski). Then we have

$$\frac{\partial}{\partial t}(SZ - (SZ)^{\mathrm{T}}) = \frac{\mathrm{D}}{\mathrm{D}t}(SZ - (SZ)^{\mathrm{T}}) - F^{\alpha}\frac{\partial}{\partial x^{\alpha}}(SZ - (SZ)^{\mathrm{T}})$$

with $x = \mathrm{col}(q, \dot{q})$, and $F = \mathrm{col}(\dot{q}, f)$ as in (18). In other words we have

$$\frac{\partial}{\partial t}(SZ - (SZ)^{\mathrm{T}}) = -F^{\alpha} \frac{\partial}{\partial x^{\alpha}}(SZ - (SZ)^{\mathrm{T}}) + G(t, x) \tag{41}$$

with $G(t, x) = SZ^{[1]} - (SZ^{[1]})^{\mathrm{T}}$.

In view of (39(1)) we have $G(t_0, x) = 0$, and hence also $\partial G/\partial x^{\alpha}(t_0, x) = 0$. Next we know that

$$\frac{\partial G}{\partial t} = -F^{\alpha} \frac{\partial G}{\partial x^{\alpha}} + H(t, x)$$

with $H = SZ^{[2]} - (SZ^{[2]})^{\mathrm{T}}$.

Using (39(2)) together with the above intermediate results thus shows that $\partial G/\partial t(t_0, x) = 0$. Proceeding in the same way we shall find $\partial^k G/\partial t^k(t_0, x) = 0$ for all k. Hence by (41), $SZ - (SZ)^{\mathrm{T}}$ satisfies a partial differential equation of the type (22), together with all further assumptions of lemma 3. We thus conclude $SZ = (SZ)^{\mathrm{T}}$, and the desired result now follows from theorem 5.

Let us extensively comment on this theorem, to show its relevance and because it seems to us that this is the end of the line, inasmuch as one does not want to break the elegant matrix formulae (39) apart. For a given system (1), all matrices $\Phi^{(k)}(t_0, q, \dot{q})$ consist of known elements. So one can start solving equations (39) algebraically and step by step for the independent elements of a symmetric matrix γ. If at a certain stage the freedom in γ, left over from previous conditions, is no longer sufficient to cover the next condition, then at least by purely algebraic manipulations one has reached the conclusion that no Lagrangian exists in the given coordinates.

Alternatively, (39) is an infinite system of homogeneous algebraic equations in the $n(n+1)/2$ independent elements of γ. Such a system can only have non-zero solutions if the rank of the corresponding coefficient matrix is at most $\frac{1}{2}n(n+1) - 1$, which by the way is smaller than in Henneaux' discussion, because of the hidden identities involved there (see Henneaux 1981). So by explicitly expressing this rank condition one can, in principle, write down some awfully complicated partial differential equations for the functions $f^i(t, q, \dot{q})$. Douglas (1941), for the case $n = 2$, combined the first of these determinant conditions with the full partial differential equations to be satisfied by α, to arrive at a complete classification of the two degrees of freedom case. The result stated in theorem 6 suggests the possibility of looking first at all the conditions to be satisfied by some time-independent matrix γ, whereby it is again of interest to point out that (apart from the lower rank condition we have obtained) we have also found a reduction of the number of closure conditions to be imposed, in comparison with the 'first-order picture' of the appendix. Such a further analysis, however, will strongly depend on the dimension of the given system. In the next section we will derive some further necessary algebraic conditions which could be helpful for such as purpose. But first let us discuss an interesting situation, in which the infinite set of conditions (39) will reduce to a finite one.

Corollary 6.1. Assume that for some k, scalar functions $\lambda_l(t, q, \dot{q})$ and $\mu(t, q, \dot{q})$ can be found such that

$$\Phi^{(k)}(t, q, \dot{q}) = \sum_{l=0}^{k-1} \lambda_l \Phi^{(l)} + \mu 1. \tag{42}$$

Then the necessary and sufficient conditions reduce to (38), and (39(l)) for $l = 0, 1, \ldots, k-1$.

Proof. Assumption (42) implies a linear dependence of all $\Phi^{(m)}$, $m \geq k$ on lower-order ones, from which the result trivially follows.

Corollary 6.2. The conclusions of corollary 6.1 hold *a fortiori* for the case that

$$\Phi^{(k)}(t, q, \dot{q}) = 0 \qquad \text{for some } k. \tag{43}$$

A condition like (43) is what we called 'kth order commutativity' in the case of linear systems. Here such conditions do not have the same degree of sufficiency for existence of a Lagrangian as they had in the linear case. Still they characterise rather special situations, as is illustrated by the property that $\Phi^{(k)} = 0$ implies that the eigenvalues of $\Phi^{(k-1)}$ will be first integrals of the given system, which is readily concluded from

$$\frac{\mathrm{D}}{\mathrm{D}t}(U^{-1}\Phi^{(k-1)}U) = 0.$$

5. Further algebraic necessary conditions

Proposition 7. If a multiplier α for (1) exists, then its initial value $\gamma(q, \dot{q})$ must necessarily satisfy the following infinite set of supplementary algebraic relations

$$\gamma_{lj}\left(\frac{\partial \phi_{jk}}{\partial \dot{q}^i} - \frac{\partial \phi_{ji}}{\partial \dot{q}^k}\right) + \gamma_{ij}\left(\frac{\partial \phi_{jl}}{\partial \dot{q}^k} - \frac{\partial \phi_{jk}}{\partial \dot{q}^l}\right) + \gamma_{kj}\left(\frac{\partial \phi_{ji}}{\partial \dot{q}^l} - \frac{\partial \phi_{jl}}{\partial \dot{q}^i}\right) = 0 \tag{44}$$

where (ϕ_{ij}) here stands for any of the matrices $\Phi^{(k)}(t_0, q, \dot{q})$.

Proof. The conditions (38a), (38b) imply that γ must be of the form

$$\gamma_{ij} = \partial^2 G / \partial \dot{q}^i \partial \dot{q}^j \qquad \text{for some function } G. \tag{45}$$

Any of the conditions (39) will therefore read like

$$\frac{\partial^2 G}{\partial \dot{q}^i \partial \dot{q}^j}\phi_{jk} = \frac{\partial^2 G}{\partial \dot{q}^k \partial \dot{q}^j}\phi_{ji}$$

which can be written in the form

$$\frac{\partial}{\partial \dot{q}^i}\left(\frac{\partial G}{\partial \dot{q}^j}\phi_{jk}\right) - \frac{\partial}{\partial \dot{q}^k}\left(\frac{\partial G}{\partial \dot{q}^j}\phi_{ji}\right) = \frac{\partial G}{\partial \dot{q}^j}\left(\frac{\partial \phi_{jk}}{\partial \dot{q}^i} - \frac{\partial \phi_{ji}}{\partial \dot{q}^k}\right). \tag{46}$$

The curl structure of the left-hand side of (46) allows the introduction of a closed 2-form. Expressing the closedness of this 2-form via the right-hand side of (46) (and using (45)) leads to the required conclusion.

The additional conditions (44) exhibit the role that the dimension will play in any further investigation going beyond the results of theorem 6. This is already apparent from looking e.g. at the case of 'first-order commutativity' $\Phi^{(1)} = 0$. Indeed, the infinite set of conditions (39) then reduces to only one symmetry requirement, which leaves an n-parameter family of possible solutions for γ. For the linear case, this immediately leads to an n-parameter family of Lagrangians. But here we still have the conditions (38b, c), which in particular lead to the supplementary algebraic relations (44) with

$\phi = \Phi^{(0)}(t_0, q, \dot{q})$. Now for $n = 2$, conditions (44) are identically satisfied. For $n = 3$, they give rise to exactly one relation, reducing the original three-parameter family of solutions for γ to a two-parameter family. For $n = 4$ we get four extra conditions, leading already to a vanishing determinant requirement. Hence, starting from $n = 4$, instead of looking solely at the rank of the algebraic system (39), it may be advantageous to draw necessary conditions on the f^i alone from vanishing determinants, resulting from (39, (0)) and (44) for $\Phi^{(0)}$. In this light, a possibly interesting case for further study would be

$$\partial\phi_{ij}/\partial\dot{q}^k = \partial\phi_{ik}/\partial\dot{q}^j \tag{47}$$

because (44) then certainly does not impose further restrictions on the algebraic freedom in γ.

6. An example

It seems to us that the main power of the infinite set of algebraic conditions lies in the possibility of arriving at a conclusion by very simple means for cases where no Lagrangian exists. Nevertheless, they will sometimes allow us also to draw positive conclusions, an example of which was given in Sarlet *et al* (1982). Once the existence of a Lagrangian is confirmed, however, the actual construction of all multipliers will often be possible by relying solely on theorem 1. We should like to illustrate this by constructing all Lagrangians for the two-dimensional Kepler problem, whose equations of motion are

$$\ddot{q}_1 = -\mu q_1/r^3 = f_1 \qquad r = (q_1^2 + q_2^2)^{1/2} \qquad \mu \text{ constant.} \tag{48a}$$
$$\ddot{q}_2 = -\mu q_2/r^3 = f_2 \tag{48b}$$

For this example we have

$$A = 0 \qquad \text{and} \qquad \Phi^{(0)} = B \text{ is symmetric}$$

so that corollary 1.2 already ensures that the unit matrix is a multiplier. We then go back to the formulation of theorem 1, and try to determine the general solution for α. The algebraic condition (13d) yields

$$\alpha_{12} = \lambda(\alpha_{11} - \alpha_{22}) \tag{49}$$

with

$$\lambda = \frac{B_{12}}{B_{11} - B_{22}} = \frac{q_1 q_2}{q_1^2 - q_2^2}.$$

Now (13c) implies that all elements of α must be first integrals. Hence, if $\alpha_{11} - \alpha_{22}$ were non-zero, (49) would require that λ be a first integral, which is clearly impossible. We therefore conclude

$$\alpha_{11} = \alpha_{22} \qquad \text{and} \qquad \alpha_{12} = 0.$$

The conditions (13b) then yield

$$\partial\alpha_{11}/\partial\dot{q}_2 = 0 \qquad \partial\alpha_{22}/\partial\dot{q}_1 = 0$$

so that $\alpha_{11} = \alpha_{22}$ should be independent of $\dot{q}$, while being a first integral of the system, and therefore can only be trivially constant. Thus, the general solution for α is obtained as

$$\alpha = c\,1 \qquad c = \text{scalar constant.}$$

In this way we have recovered along different lines that the Lagrangian for the two-dimensional Kepler problem is unique up to trivialities such as gauge terms and multiplication by a constant factor (see Henneaux 1981).

7. Conclusions

All the principal results of this paper have been amply discussed throughout the text, so we content ourselves here with a brief survey.

The reduction of the Helmholtz conditions, which we originally derived for linear systems (Sarlet 1980, Sarlet *et al* 1982) has been completely carried over to the general inverse problem of Lagrangian dynamics. A first step is reflected by theorem 1. Further progress was possible by replacing some of the requirements involved by similar requirements at $t = t_0$ (assuming analyticity of the given functions f^i). In carrying this through, it was useful to know about the equivalent 'first-order picture' described in § 2, with respect to which we have obtained considerable simplifications. In some very restrictive sense one could say that the inverse problem is not solvable. By that we mean that it is impossible to arrive at a general formula for the functions f^i, which would tell us how the forces must look in order that a multiplier matrix α be constructable. By theorem 6, however, we come as close as possible to the solution, without ripping apart the still manageable form of the conditions involved. Beyond theorem 6, one will have to start discussing the rank of certain coefficient matrices, the way it was done e.g. by Douglas (1941) for $n = 2$. We have given certain indications for possible future studies along this line.

We should like to end with some rather philosophical remarks about the practical applicability of the inverse problem methodology. It is sometimes argued that the search for a Lagrangian for a general second-order system is a valuable first step to pursue, because knowledge of L (through the classical methods of Lagrangian and Hamiltonian mechanics) might subsequently be of considerable help in trying to solve the given equations. This may well be true in simple situations, such as in the case of linear systems, but it should not be emphasised too much for general nonlinear systems. Indeed, when physical arguments cannot immediately give us a Lagrangian, we have learned that constructing one may well require, either implicitly or explicitly, the determination of a whole matrix of first integrals (see the structure (29*b*) of the multiplier α), and this in turn could be as difficult as trying to solve the given differential equation. Nevertheless, we have been discussing here a well defined problem, with a long history of interest, and as such there is no doubt that there is room for valuable future contributions.

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Appendix. Henneaux' procedure

For convenience, here we introduce the following notational conventions: Greek indices run from 1 to $2n$; Latin indices on variables or functions belonging to a set with $2n$ elements will run from $n+1$ to $2n$; the remaining elements of this set will be indicated by Latin indices with a hat. So we have in agreement with (18)

$$x^i = \dot{q}^i \qquad x^{\hat{i}} = q^i \qquad F^i = f^i \qquad F^{\hat{i}} = \dot{q}^i$$

while the conditions (19) simply read $C_{ij} = 0$.

Note also that by using (16), one can rewrite (17) as

$$\frac{\mathrm{D}C_{\mu\nu}}{\mathrm{D}t} \equiv \frac{\partial C_{\mu\nu}}{\partial t} + \frac{\partial C_{\mu\nu}}{\partial x^\alpha} F^\alpha = C_{\nu\alpha} \frac{\partial F^\alpha}{\partial x^\mu} - C_{\mu\alpha} \frac{\partial F^\alpha}{\partial x^\nu}. \tag{A1}$$

In a slight adaptation of Henneaux' reasoning we can find an infinite set of necessary conditions by computing consecutive total time derivatives of (19) and making use of (A1). The first two steps lead to

$$C_{j\hat{i}} - C_{i\hat{j}} = 0 \qquad 2C_{\hat{i}\hat{j}} = C_{\hat{j}k} \frac{\partial f^k}{\partial \dot{q}^i} - C_{\hat{i}k} \frac{\partial f^k}{\partial \dot{q}^j}. \tag{A2}$$

While the first of these implies the symmetry of α in the representation (20) of C, the second condition shows that β and α indeed must be interrelated through (9b).

After going through § 4, the reader will have no difficulty in comparing the infinite set of conditions we shall derive there with the conditions which would follow in the 'first-order picture' by pushing the procedure beyond step (A2).

Let us finally show here that the closure conditions (21) are indeed identically satisfied. Making use of (9a) and (6), we obtain (CP means cyclic permutation)

$$\begin{aligned} \frac{\partial \beta_{ij}}{\partial q^k} + \mathrm{CP} &= \frac{1}{2} \frac{\partial}{\partial q^k} \left(\frac{\partial \beta_j}{\partial \dot{q}^i} - \frac{\partial \beta_i}{\partial \dot{q}^j} \right) + \mathrm{CP} = \frac{1}{2} \frac{\partial}{\partial \dot{q}^k} \left(\frac{\partial \beta_i}{\partial q^j} - \frac{\partial \beta_j}{\partial q^i} \right) + \mathrm{CP} \\ &= \frac{1}{4} \left[\frac{\partial}{\partial q^k} \left(\frac{\partial \beta_i}{\partial \dot{q}^j} - \frac{\partial \beta_j}{\partial \dot{q}^i} \right) + \mathrm{CP} \right] + \frac{1}{4} \left(\frac{\partial}{\partial t} + \dot{q}^l \frac{\partial}{\partial q^l} \right) \left[\frac{\partial}{\partial \dot{q}^k} \left(\frac{\partial \beta_i}{\partial \dot{q}^j} - \frac{\partial \beta_j}{\partial \dot{q}^i} \right) + \mathrm{CP} \right] \end{aligned} \tag{A3}$$

from which the result readily follows.

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